just the opposite of the situation when the work of Thomas and Young is considered.

The discrepancies among the various investigators are outside the limits of estimated experimental uncertainty. Critical analysis and selection of the most reliable data are in the province of another laboratory, where the system is now under study (*2*).

ACKNOWLEDGMENT

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Control-System Design for a Chemical Process by the Root-Locus Method

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This paper illustrates the application of the root-locus method in the design of a control system for a theoretical stirred-tank reactor. The merits of control by measuring reactor concentration or temperature were considered at both an unstable and stable steady state reactor condition. The modes of control studied were proportional, proportional-integral, and proportional-integral-rate.

The primary concern in the design of all control systems is the time response of a given system to a prescribed input or set of inputs. The dynamics of a system is analyzed experimentally and/or analytically by one or a combination of several methods. Most of the standard methods utilize the Laplace transform since it transforms a differential equation in time into an algebraic equation in S, and in addition the transform permits the consideration of analysis and design problems in terms of the complex frequency $S = \sigma + i\omega$, rather than in terms of the time functions.

The frequency-response analysis is the conventional method of obtaining time-response information. It provides such information as system stability, frequency of damped oscillation, damping, and steady state error. The method is simple to apply, and the frequency response of a component or system may be obtained mathematically or by laboratory measurement. The inherent reasons for the simplicity of this method, however, result in limiting its usefulness. At the outset of a design the complex frequency S is replaced only by $i\omega$, and thus the analysis is narrowed to the $i\omega$ axis of the S plane. Since the real part of S, (σ), is not considered, the opportunity of maintaining control over both frequency and transient responses is discarded.

The more recent design methods, such as the root-locus (*1*) and Guilleman's (*1*) methods, take into account the entire S plane. This is achieved by working through the poles and zeros of various transfer functions and thus simultaneously controlling the frequency and transient responses. A detailed description of the root-locus method (which is used in this paper) may be found in most recent texts on automatic control (*1*).

It should be pointed out that the control-system design is not for any specific stirred-tank chemical reactor but is concerned with the fundamental variables of a theoretical stirred-tank reactor. The calculated results have not been verified experimentally.

The basic continuous stirred-tank reactor process used in this paper was originally described by Aris and Amundson (*2*).

GENERAL CONTROL PROBELM

Description of Control-System Process

The process to be controlled is the continuous stirred-tank chemical reactor shown in Figure 1. The influent consists of a reactant A and an inert carrier. The reaction occurring in the reactor is

$$2A \longrightarrow \text{products}$$

The reaction is assumed to be exothermic, irreversible, and second order. The material balance around the reactor may be written

$$\frac{VdA}{dt} = Q(A_0 - A) - kVA^2 \quad (1)$$

Equation (1) states that the rate of accumulation of A in the reactor is equal to the difference between the amount of A entering minus the amount of A leaving and the amount that has disappeared owing to the reaction.

The energy balance is

$$Vc\rho \frac{dT}{dt} = Qc\rho(T_0 - T) - ha(T - T_c) + kVA^2(-\Delta H) \quad (2)$$

Thus the accumulation of heat in the reactor is equal to the difference in the total heat removed (by warming the influent and by the cooling coil) and the heat generated by the reaction.

The average coolant temperature and the inlet coolant temperature are related by

$$T_c = \frac{(\alpha/Q_c)T + T_{ec}}{(\alpha/Q_c) + 1} \quad (3)$$

where α is defined as

$$\alpha = \frac{ha}{2c_c\rho_c} \quad (4)$$

Equation (5) is then obtained by substituting Equations (3) and (4) into Equation (2):

$$Vc\rho \frac{dT}{dt} = Qc\rho(T_0 - T) - \frac{Q_c ha}{\alpha + Q_c} \cdot (T - T_{ec}) - kVA^2(\Delta H) \quad (5)$$

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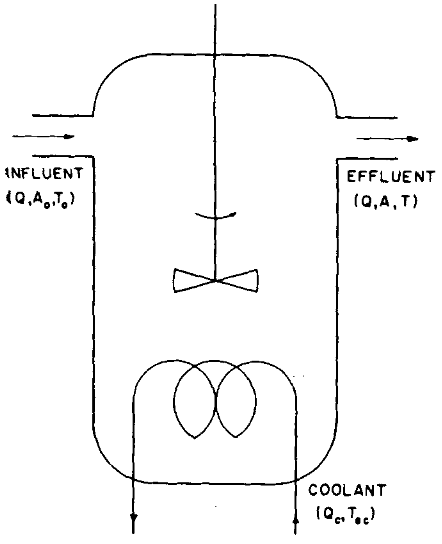

Fig. 1. Continuous stirred-tank reactor.

Also the reaction velocity coefficient may be represented by

$$k = k_1 e^{-\epsilon/RT} \tag{6}$$

It has been shown (*3*) that if these equations are solved in terms of A or T with steady state conditions being assumed (that is, dT/dt and dA/dt equal zero), then there may exist one to three solutions depending upon the values of the equation parameters. For the case of three solutions there will be three sets of steady state values of T and A. The high and low values of T are stable steady states, and the intermediate value represents an unstable steady state. The physical behavior of these steady state locations may be explained as follows. If the reactor is being operated at either the upper or lower steady state and a perturbation in reactor temperature occurs, the reactor will tend to adjust itself back to the original steady state value. If the temperature increases slightly, the rate of heat removed by the cooling coil is greater than that generated by the reaction and the reactor cools to the original steady state. At the unstable steady state however a temperature increase will cause the rate of heat generation to be greater than the rate removed, and the reactor will thus adjust to a higher temperature. The reverse is true if the reactor temperature is slightly decreased from the unstable steady state temperature.

Therefore to obtain an over-all satisfactory control of the reactor special attention must be centered on obtaining proper control at an unstable steady state. In the following sections both the unstable and stable steady state conditions will be considered, and each will be separately analyzed with concentration and temperature as the directly controlled variables.

Derivation of Process Equation

The solution of the basic Equations (1), (5), and (6) in terms of an explicit variable A or T would be difficult if not impossible. A simplification of Equations (1) and (5) may be effected by linearizing the nonlinear terms. This may be accomplished by expanding each nonlinear term in a Taylor's series and retaining only the linear terms. The Taylor's series for a function of two variables may be written as

$$f(x, y) = f(x_0 y_0) + \left(\frac{\partial f}{\partial x}\right)_{x_0 y_0} (x - x_0) + \left(\frac{\partial f}{\partial y}\right)_{x_0 y_0} (y - y_0) + \cdots$$

Since the variables in Equations (1) and (5) are A, A_0, T, T_0, T_{ec}, and Q_c, the following terms must be linearized:

$$kA^2; \quad \frac{Q_c haT}{\alpha + Q_c}; \quad \frac{Q_c haT_{ec}}{\alpha + Q_c}$$

The linearization of kA^2, for example, is

$$kA^2 = k_1 e^{-\epsilon/RT} A^2$$
$$= k_s A_s^2 + 2k_s A_s (A - A_s) + k_s A_s^2 \frac{\epsilon}{RT_s^2} (T - T_s)$$

The expansion is about a steady state value, and these constants will be denoted by a subscript, s.

With Equation (6) substituted into (1) and (5) these equations then become

$$\frac{dA}{dt} = -\left(\frac{Q_s}{V} + 2k_s A_s\right)A - \left(\frac{k_s A_s^2 \epsilon}{RT_s^2}\right)T + \left(\frac{Q_s}{V}\right)A_0 + k_s A_s^2 + \frac{k_s A_s^2 \epsilon}{RT_s} \tag{7}$$

$$\frac{dT}{dt} = -\left(\frac{2\Delta H k_s A_s}{c\rho}\right)A - \left(Q_s + \frac{\Delta H k_s A_s^2 \epsilon}{c\rho R T_s^2} + \frac{Q_{cs} ha}{Vc\rho(\alpha + Q_{cs})}\right)T + \left(\frac{Q}{V}\right)T_0 + \left(\frac{Q_{cs} ha}{Vc\rho(\alpha + Q_{cs})}\right)T_{ec} - \left(\frac{\alpha ha(T_s - T_{ecs})}{Vc\rho(\alpha + Q_{cs})^2}\right)Q_c + \frac{\Delta H k_s A_s^2 \epsilon}{c\rho R T_s} + \left(\frac{\alpha ha(T_s - T_{ecs})}{Vc\rho(\alpha + Q_{cs})^2}\right)Q_{cs} \tag{8}$$

Values of the constants in Equations (7) and (8) are chosen specifically to produce either a stable or unstable steady state reactor. The constants are reasonable values but are not for any real system.

Upon substitution of the values, solution of the equations for A or T, and transformation, Equations (9), (10), (11), and (12) result.

For the unstable steady state

$$\mathbf{A} = \frac{0.005(S - 0.0159)\mathbf{A}_0 - 3.39\times10^{-7}\mathbf{T}_0 - 1.13\times10^{-7}\mathbf{T}_{ec} + 1.97\times10^{-5}\mathbf{Q}_c + (0.704S^2 - 9.75\times10^{-3}S + 2.88\times10^{-4})(1/S)}{(S - 0.0113)(S + 0.00457)} \tag{9}$$

$$\mathbf{T} = \frac{7.0\times10^{-3}\mathbf{A}_0 + (0.005\mathbf{T}_0 + 0.00167\mathbf{T}_{ec} + 0.290\mathbf{Q}_c)(S + 9.20\times10^{-3})}{(S - 0.0113)(S + 0.00457)} + \text{constant} \tag{10}$$

For the stable steady state

$$\mathbf{A} = \frac{0.005(S - 0.0410)\mathbf{A}_0 - 7.16\times10^{-7}\mathbf{T}_0 - 2.39\times10^{-7}\mathbf{T}_{ec} + 7.83\times10^{-5}\mathbf{Q}_c + (0.0589S^2 + 0.00228S + 0.000160)(1/S)}{(S + 0.116)(S + 0.00740)} \tag{11}$$

$$\mathbf{T} = \frac{0.266\mathbf{A}_0 + (0.005S + 8.23\times10^{-4})\mathbf{T}_0 + (0.00167S + 2.75\times10^{-4})\mathbf{T}_{ec} - (0.547S + 0.090)\mathbf{Q}_c}{(S + 0.116)(S + 0.00740)} + \text{constant} \tag{12}$$

The constant input terms in Equations (10) and (12) (last terms), represent the steady state values of A and T. This may be shown by the application of the initial-value theorem, which gives the value of the function at time = 0. Therefore these terms may be neglected and the steady state value then added to the ordinate of all transient response curves.

In the following sections it will be shown that the control of this process essentially consists of maintaining the system at some predetermined steady state value by regulating it against variations due to changes in A_0, T_0, and T_{ec}. (Disturbances in Q_c are assumed negligible.)

No time lags that would be introduced into the control loop by the actual operation of the control equipment (values, sensing elements, etc.) will be considered in this analysis.

Block Diagram

The variables $\mathbf{A}_0$, $\mathbf{T}_0$, $\mathbf{T}_{ec}$, and $\mathbf{Q}_c$ in Equations (9) and (11) may be considered as signals which are operated on by their transfer functions and contribute accordingly to the value of **A**. The same is true for Equations (10) and (12) where **T** is the measured variable. The control of the reactor at some specified value of A or T may be accomplished by a single feedback loop regulation of one or more of the variables $\mathbf{A}_0$, $\mathbf{T}_0$, $\mathbf{T}_{ec}$, and $\mathbf{Q}_c$.

The block diagram for the case in which the controller regulates the flow of water through the cooling coil is shown in Figure 2. The desired concentration **A** or temperature **T** in the reactor is indicated by $\boldsymbol{\theta}_i$; the measured variable **A** or **T** is denoted by $\boldsymbol{\theta}_0$. If a signal is entered into the system by a change in one or a combination of the variables $\mathbf{A}_0$, $\mathbf{T}_{ec}$, $\mathbf{T}_0$, or $\mathbf{Q}_c$, each signal is operated on by the transfer function of the block and contributes to the value of the output variable. This signal is fed back to the comparitor (usually part of the controller) where it is compared with the desired variable $\boldsymbol{\theta}_i$. The difference in signals (error signal) then goes to the controller which acts to correct the error by readjusting the flow in the cooling coil. In actual operation $\boldsymbol{\theta}_i$ is held constant for long periods of time, so that the response specification for control over changes in $\boldsymbol{\theta}_i$ may be within a very liberal range. The dotted lines represent signals within the reactor that cannot be measured; that is it is not possible in the physical system to isolate any concentration or temperature measurement as being due to $\mathbf{A}_0$, $\mathbf{T}_{ec}$, $\mathbf{T}_0$, or $\mathbf{Q}_c$. It is convenient however to consider the signals as isolated; as will be shown later, this assumption offers no significant limitation in the control system design. The single-loop feedback type of control system (Figure 2) will be used throughout this paper. It is by no means the only or possibly even the best system to use in controlling this process; it is practical, however, since it is simple and widely used in industry today. It should therefore be analyzed first as a possible suitable control system. Other systems might include a minor feedback loop to the node at $\mathbf{Q}_c$, feed forward control of $\mathbf{A}_0$ (*4*), and combinations of these with concentration and/or temperature being the measured variables. An equation representing the block diagram in Figure 2 may be derived as follows. From Figure 2 it is seen that

$$\boldsymbol{\theta}_0 = \boldsymbol{\theta}_i - E \qquad (13)$$

and

$$E = \frac{\boldsymbol{\theta}_0}{G} \qquad (14)$$

where

$$\boldsymbol{\theta}_0 = GE + L_{T_0}\mathbf{T}_0 + L_{T_{ec}}\mathbf{T}_{ec} + L_{A_0}\mathbf{A}_0 + N(1)$$

and

$$G = G_{EQ_c}G_{Q_c}$$

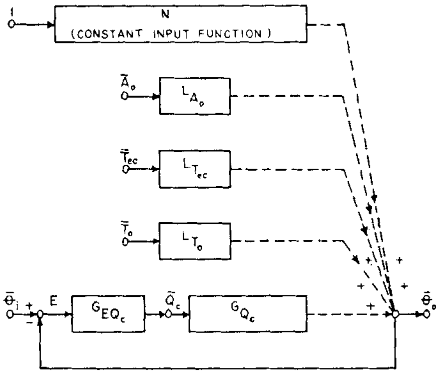

Fig. 2. Block diagram for control of reactor coolant.

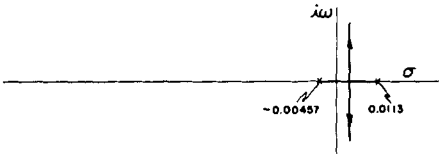

Fig. 3. Root loci for Equation (22).

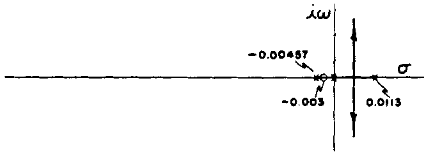

Fig. 4. Root loci for Equation (24) with $z_1 = 0.003$.

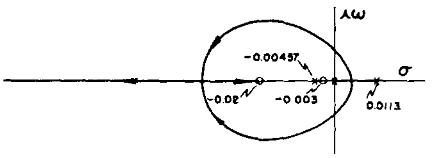

Fig. 5. Root loci for Equation (26) with $z_1 = 0.003$ and $z_2 = 0.02$.

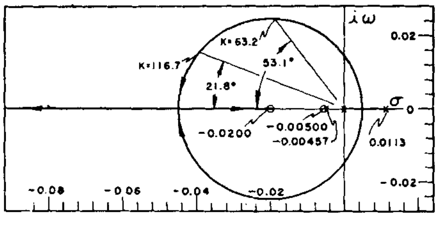

Fig. 6. Root loci for Equation (26) with $z_1 = 0.005$ and $z_2 = 0.020$.

Therefore the block diagram is represented by

$$\boldsymbol{\theta}_0 = \frac{G}{1+G}\boldsymbol{\theta}_i + \frac{L_{T_0}}{1+G}\mathbf{T}_0 + \frac{L_{T_{ec}}}{1+G}\mathbf{T}_{ec} + \frac{L_{A_0}}{1+G}\mathbf{A}_0 + \frac{N(1)}{1+G} \qquad (15)$$

DESIGN IN TERMS OF ROOT LOCI

Basic Principles of the Root-Locus Method

The root-locus method is a graphical technique for the determination of the roots of a polynomial. It has a special application in the analysis of feedback control systems, however, since unique functional relationships exist in these systems.

The basic elements of the root-locus method may be illustrated by consideration of the system in Figure 2. If for the sake of simplicity one assumes that there are no disturbances in the system, Equation (15) (the closed-loop system function) may be written as

$$\frac{\boldsymbol{\theta}_0}{\boldsymbol{\theta}_i} = \frac{G}{1+G} \qquad (16)$$

Since G is usually a ratio of rational algebraic functions, it may be represented by two polynomials in S:

$$G = \frac{p(S)}{q(S)} \qquad (17)$$

Substitution of Equation (17) in (16) gives

$$\frac{\boldsymbol{\theta}_0}{\boldsymbol{\theta}_i} = \frac{\dfrac{p(S)}{q(S)}}{1 + \dfrac{p(S)}{q(S)}} = \frac{p(S)}{p(S) + q(S)} \qquad (18)$$

Thus since the zeros of $p(S)$ and $q(S)$ are known, the root-locus method is a graphical technique for determining the zeros of $p(S) + q(S)$ from the zeros of $p(S)$ and $q(S)$ individually.

Also since G contains the variable gain factor, the root loci then are the roots of the equation $G = p(S)/q(S) = -1$, as a function of the gain. Further discussion of this material and rules for construction of the root loci may be found in standard texts (*1, 5*).

Unstable Steady State

Concentration as the Measured Variable

The process for this condition is represented by Equation (9). As mentioned previously, control will be maintained through the regulation of flow in the cooling coil. This will be true for all systems studied in this paper.

The block diagram for $\boldsymbol{\theta}_i = \mathbf{A}_i$ and $\boldsymbol{\theta}_0 = \mathbf{A}_0$ is then represented by rewriting Equation (21) (closed-loop equation) to give

$$\mathbf{A} = \frac{G}{1+G}\mathbf{A}_i + \frac{L_{T_o}}{1+G}\mathbf{T}_0 + \frac{L_{T_{ec}}}{1+G}\mathbf{T}_{ec} + \frac{L_{A_o}}{1+G}\mathbf{A}_0 + \frac{N(1)}{1+G} \qquad (19)$$

where

$$G = G_{EQ_c}G_{Q_c} = \frac{\mathbf{A}}{\mathbf{E}} \qquad (19a)$$

$$= G_{EQ_c}\frac{1.97 \times 10^{-5}}{(S - 0.0113)(S + 0.00457)}$$

$$L_{T_o} = \frac{\mathbf{A}}{\mathbf{T}_0}$$

$$= \frac{-3.39 \times 10^{-7}}{(S - 0.0113)(S + 0.00457)} \qquad (19b)$$

$$L_{T_{ec}} = \frac{\mathbf{A}}{\mathbf{T}_{ec}}$$

$$= \frac{-1.13 \times 10^{-7}}{(S - 0.0113)(S + 0.00457)} \qquad (19c)$$

$$L_{A_o} = \frac{\mathbf{A}}{\mathbf{A}_0}$$

$$= \frac{0.005(S - 0.0159)}{(S - 0.0113)(S + 0.00457)} \qquad (19d)$$

$$N = \frac{\mathbf{A}}{1} \qquad (19e)$$

$$= \frac{0.704S^2 - 9.75\times10^{-3}S + 2.88\times10^{-4}}{S(S-0.0113)(S+0.00457)}$$

The characteristic equation for which the root loci are to be calculated is the polynomial $1 + G = 0$, or $G = -1$. This will be solved for various controller functions, G_{EQ}, as a function of the control gain.

The root loci of $G = -1$ are the S-plane points at which

$$\angle G = \tan^{-1} 0/-1$$

$$= 180° + n360° \qquad (20)$$

This equation is the basis for all rules and techniques for construction of the root loci.

Proportional Control

The controller equation for proportional control is given by

$$G_{EQ_c} = K \qquad (21)$$

The open-loop equation is then

$$G = G_{EQ_c}G_{Q_c}$$

$$= \frac{K1.97 \times 10^{-5}}{(S - 0.0113)(S + 0.00457)} \qquad (22)$$

The root-locus plot for Equation (22) is shown in Figure 3. It is obvious from the location of the loci that the roots

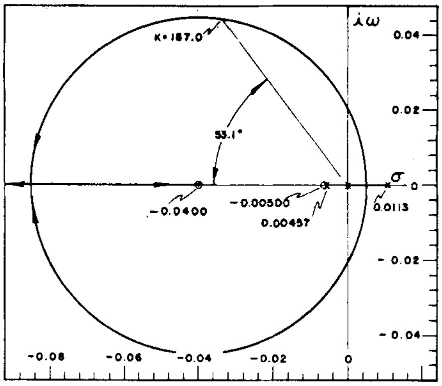

Fig. 7. Root loci for Equation (26) with $z_1 = 0.005$ and $z_2 = 0.040$.

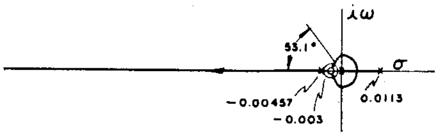

Fig. 8. Root loci for Equation (26) with $z_1 = z_2 = 0.003$.

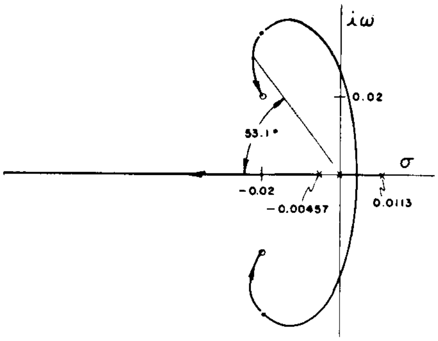

Fig. 9. Root loci for Equation (26) with $z_1 = 0.02 + i0.02$ and $z_2 = 0.02 - i0.02$.

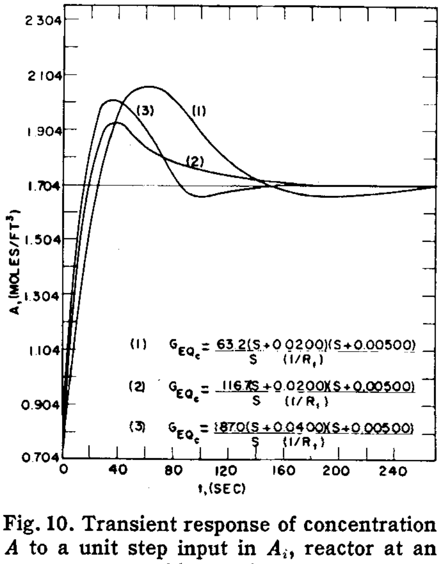

Fig. 10. Transient response of concentration A to a unit step input in A_i, reactor at an unstable steady state.

are positive, and the system will therefore be unstable for all values of K.

Proportional-Integral Control

The equation for the controller is

$$G_{EQ_c} = \frac{K}{S}\left(S + \frac{1}{I_t}\right)$$

$$= \frac{K}{S}(S + z_1) \qquad (23)$$

The open-loop equation may then be written as

$$G = \frac{K1.97 \times 10^{-5}(S + z_1)}{S(S - 0.0113)(S + 0.00457)} \qquad (24)$$

The root loci for G in Equation (24) (with $z = 0.003$) are shown in Figure 4. Two of the roots are positive for all values of K, and thus this system will always be unstable.

Proportional-Integral-Rate Control

The equation for the controller is given by

$$G_{EQ_c} = \frac{KR_t}{S}\left(S^2 + \frac{S}{R_t} + \frac{1}{I_tR_t}\right)$$

$$= \frac{KR_t}{S}(S + z_1)(S + z_2) \qquad (25)$$

The open-loop equation is therefore

$$G = \frac{KR_t1.97\times10^{-5}(S+z_1)(S+z_2)}{S(S-0.0113)(S+0.00457)} \qquad (26)$$

For $z_1 = 0.003$ and $z_2 = 0.02$ the root loci are sketched in Figure 5 (without any regard for accuracy). In this case it is evident that the system will be stable for all values of K for which the loci are on the negative side of the $i\omega$ axis. Thus it is seen that the addition of the rate action is necessary to attain system stability. Other root-locus diagrams may be constructed for the proportional-integral-rate action by moving the zeros of G to different locations in the S plane. Some of these loci curves are shown in Figures 6 to 9. The loci of Figures 6 and 7 are accurate; those of Figure 8 are not accurate, and those of Figure 9 are accurate only at the points indicated on the loci curve. System stability may be attained with each of these loci configurations.

Response Specifications

The problem now arises as to which of the infinite number of possible root-locus diagrams should be used as the basis for the control system design. First of all, since no specific response specifications are known, only the relative effect of the change in a zero location can be analyzed. This does not limit the usefulness of the design, however, since optimum specifications for various systems are known. Thus the control system design is concerned with approaching the optimum specifications.

Complete control over the effects of disturbances is ordinarily achieved through a transient response analysis, since simple correlations between the frequency and time domain are usually not available. Design by shaping the frequency-response

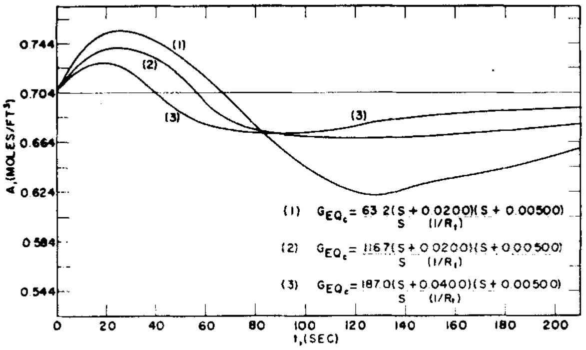

Fig. 11. Transient response of concentration A to a unit step input in A_0, reactor at an unstable steady state.

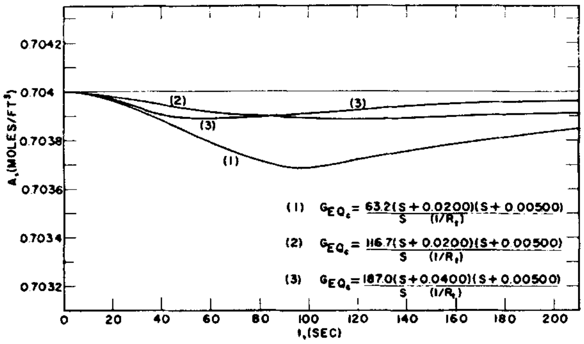

Fig. 12. Transient response of concentration A to a unit step input in T_0, reactor at an unstable steady state.

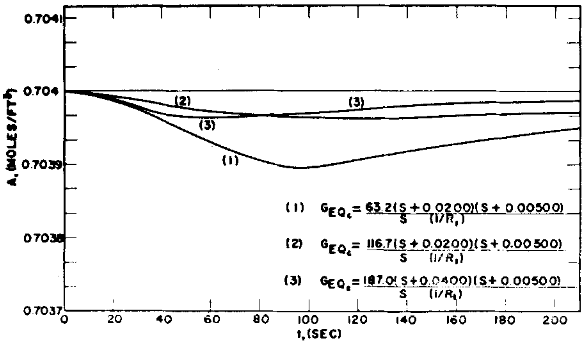

Fig. 13. Transient response of concentration A to a unit step input in T_{ci}, reactor at an unstable steady state.

curve is more applicable in control-point changes. The time-domain specifications commonly include the following three characteristics of the response of the system to a unit-step function of input: (1) the ultimate state or steady state value, (2) the settling time (or longest significant time constant), and (3) the maximum overshoot. The first quantity measures the steady state error, which is the difference between the steady state value and a constant input signal; the second measures the speed at which the system settles down after a control-point change or a disturbance, and the third measures the relative stability. The frequency-response specifications vary according to whether control is to be mainly over a disturbance or a control-point change. In either case a compromise must be made, since satisfying the specifications for a disturbance usually results in producing a less favorable response for a control-point change. The reverse is likewise true.

The optimum system for control of a disturbance is one in which the ratio of the output signal to each disturbance input signal is equal to zero. This ideal situation can be realized if the controller gain is infinite. Therefore in systems in which both reference input changes and disturbances occur the design attempts to realize a high gain while at the same time ensuring suitable relative stability and/or freedom from noise interference. An attempt at this compromise in gain usually involves the proper choice of the controller function.

Interpretation of the Root Loci

The relative merits of the various root-locus diagrams may now be analyzed with regard to satisfying the general response specifications. For the control of a continuous stirred-tank reactor it is evident that the main problem is in minimizing the effect of disturbances. The effect of reference input changes will also be analyzed, since the physical operation of a stirred-tank reactor would at times entail a change in the control point (during a start-up, etc.).

When one again refers to the root-locus diagrams (Figures 3 to 9), it is obvious that any further consideration of Figures 3 and 4 is unnecessary, since the system will be unstable for all values of K. Figures 5 to 9 on the other hand should be analyzed further to determine which would give the most desirable system response. The same scale was used in constructing these plots in order to facilitate a visual comparison of the vector magnitudes. Figures 5, 6, and 7 are of the same general type, all exhibiting possibilities of obtaining a suitable transient and frequency response. In each of these plots the gain may be increased to cause an increase in stability. The settling time is determined by the closed-loop pole nearest the $i\omega$ axis if the pole has a significant residue magnitude. For disturbances this pole is seen to be approximately -0.005. The root-locus plot in Figure 8 likewise indicates that increasing K will increase the relative stability; however the over-all stability is less, owing to the large amplitude of the cyclic part of the time response. Also the settling time for this design will be longer since the complex poles are closer to the $i\omega$ axis (with a significant residue in the complex pole) than the closest significant pole in the previous loci plots. If the loci in Figure 9 were used as the basis for design, the range over which K may be moved to adjust the response characteristics is limited by the zero locations. Hence a system with high damping could not be achieved.

The root-locus diagrams just discussed were intended to represent a rough distribution of the infinite number of possible configurations. From the preliminary analysis of these loci plots it is now reasonable to assume that the most suitable control-system design should be based on a root-locus plot similar to those shown in Figures 5, 6, and 7.

Transient Response

The transient response was obtained by introducing a unit-step function into Equation (19) and then performing the inverse transformation of this equation. Figures 10 to 13 are the results obtained for root-loci values corresponding to a ζ of 0.6 ($K = 63.2$) and 0.929 ($K = 116.7$) in Figure 6, and for a ζ of 0.6 ($K = 187.0$) in Figure 7.

From a comparison of the transient response curves the following conclusions are apparent:

1. An increase in K from 63.2 to 116.7,

as described by response curves 1 and 2 in Figures 10 to 13, causes a decrease in the settling time for both a change in reference input ($\mathbf{A}_i$) and for all the disturbance inputs ($\mathbf{T}_0$, $\mathbf{T}_{ec}$, and $\mathbf{A}_0$). An additional increase in K would further minimize the disturbance inputs. This desirable effect would however become offset with continued increases in K, since the controller would have become sensitive to extraneous (noise) signals.

2. The maximum overshoot decreases with an increase in K. This is observed for a reference-input change as well as for all disturbance inputs and may be interpreted as an increase in relative stability.

3. In all cases the principal effect of moving the zero from -0.0200 to -0.0400 (response-curve 3) is to decrease the settling time. For the disturbance-response curves the result is due to a decrease in magnitude of the residue in the dominant pole. This effect for a change in A_i is due to an increase in damping of the dominant complex poles.

4. The overshoot of curve 3 is larger than curve 2 for a step change in A_i, which means that curve 3 represents a less stable response than curve 2. The relative stability of these two response curves in the case of disturbance inputs (Figures 11, 12, and 13) is approximately the same.

Realistic Inputs

The unit-step input just discussed is obviously not of a realistic magnitude. If a step input of a different magnitude were introduced into the variables, the only effect would be to change the scale of the ordinate (A). Therefore any point on the unit-step input response curve can be interpreted as the percentage change in A (from A_s) at the corresponding lapsed time.

In order that the relative effect of each disturbance might be easily seen, the ordinate of curve 1 in Figures 11, 12, and 13 was changed by introducing the following inputs into Equation (19):

Variable	Input
A_0	0.03 lb.-mole/cu. ft.
T_0	10.0°R.
T_{ec}	5.0°R.

These input values are more realistic, and it has been shown (*4*) that disturbances of this order of magnitude are sufficiently small so that their effects may be described by the linearized equation. A plot of the transient response for each disturbance is shown in Figure 14. From a comparison of these curves it is reasonable to generalize that under actual conditions of reactor operation disturbances in A_0 and T_0 would be the most difficult to control. If some or all of the disturbances should occur simultaneously, the effect may be easily seen by a visual summation of the curves.

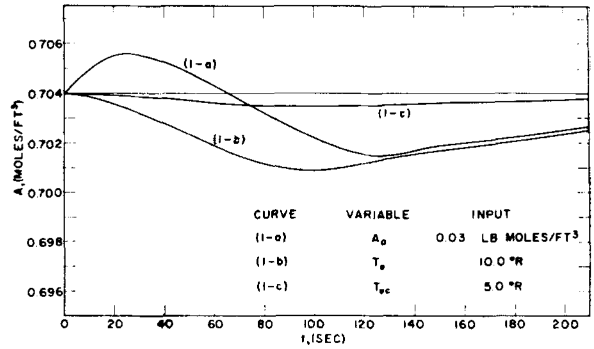

Fig. 14. Transient response of concentration A to various step input disturbances, reactor at an unstable steady state.

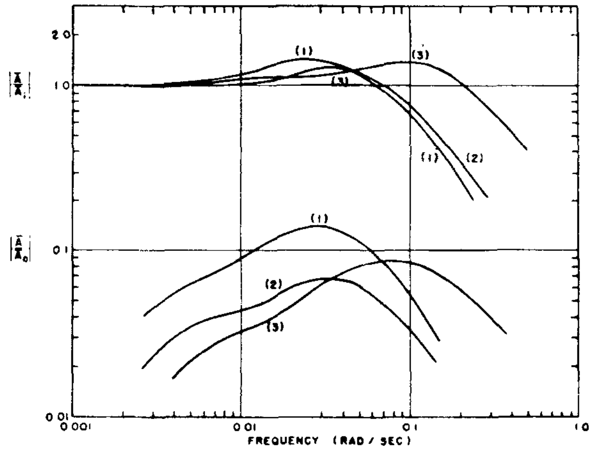

Fig. 15. Closed-loop gain characteristic of concentration, reaction at an unstable steady state.

Frequency Response

The closed-loop gain characteristics of $|\mathbf{A}/\mathbf{A}_i|$ and $|\mathbf{A}/\mathbf{A}_0|$ are shown in Figure 15. The numbers on the curves refer to the associated transient-response curves in Figures 10 and 11. The interpretation of the shapes of the curves for the frequency response of $\mathbf{A}/\mathbf{A}_i$, in terms of the expected transient response characteristics, is seen to correspond to the results shown in Figure 10. The frequency-response curves of $\mathbf{A}/\mathbf{A}_0$ however show only a correlation and do not have a simple transient-response interpretation.

OTHER REACTOR CONDITIONS

Stable Steady State

Concentration as The Measured Variable

The problem of controlling the reactor at a stable steady state should be relatively easy, since this kind of stability is autoregulatory. Under these conditions however the problem still amounts to a control of disturbances in A_0, T_0, and T_{ec} (disturbances in Q_c considered negligible). It has been shown (*3*) that a disturbance from a stable steady state, although self-correcting, theoretically takes an infinite amount of time to come back to the steady state, the speed decreasing as the steady state is approached. With no control it can be shown that the settling time is described by

$$Me^{-0.00740t}$$

Thus the aid of a control system is necessary to reduce this long settling time.

The root-locus curve for proportional control is shown in Figure 16. It is evident from the location of the loci that the system will be stable for all values of K. Additional root-locus diagrams were constructed for the other modes of control, and a transient- and frequency-response analysis was made for the case of proportional-integral control. From the over-all analysis it was apparent that this process condition was decidedly easier to control than the unstable steady state condition.

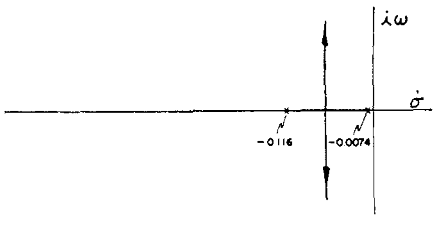

Fig. 16. Root loci for proportional control of concentration, reactor at a stable steady state.

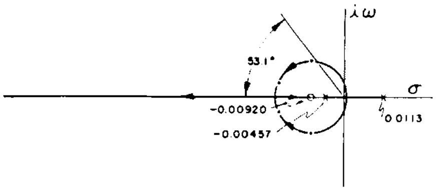

Fig. 17. Root loci for proportional control of temperature, reactor at an unstable steady state.

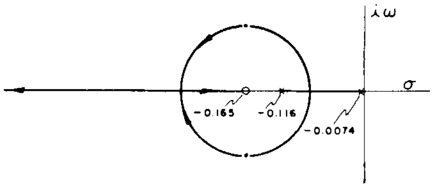

Fig. 18. Root loci for proportional control of temperature, reactor at a stable steady state.

Temperature as the Measured Variable

The unstable and stable steady state process conditions are now represented by Equations (10) and (12), respectively.

Root-locus plots were constructed for the three modes of control, of which only the plots for proportional control are shown (Figures 17 and 18). From a comparison of these two diagrams with those for concentration as the measured variable (Figures 3 and 16), it is seen that stable control of reactor temperature is easily attained at the unstable steady state; as previously explained, stable control is not possible when concentration is the measured variable (Figure 3). At the stable steady state for control of temperature (or concentration), the system will be stable for all values of K.

Introducing the other modes of control was seen to provide additional flexibility in shaping the root-locus curves, thus making possible more desirable control.

SUMMARY

The control system for a theoretical, continuous stirred-tank reactor was designed by the use of the root-locus method. The root-locus method is a well established procedure in the servomechanism field (*5*). Application of this method to process-control systems has been shown to be a valuable aid. The process was approximated by a linear system which has been shown (*4*) to be a good approximation for small disturbances about the steady state. The design was carried out for stable and unstable steady state reactor conditions. Also each condition was examined in terms of both concentration and temperature as the controlled variables.

The methods of control considered were the proportional, the proportional-integral, and the proportional-integral-rate. Numerous controller settings were examined, some of which were analyzed in detail be calculating the transient and frequency responses of the systems.

The root-locus method proved to be exceptionally useful in providing a rapid means of obtaining a picture of system behavior. A preliminary study of a large variety of systems could be made in a very short time from sketches of the approximate root loci.

To control the system at an unstable steady state, with concentration as the measured variable, proportional-integral-rate was necessary. At the stable steady state any one of the three modes of control could be used; however proportional-integral-rate action was the most desirable.

With temperature as the measured variable both the unstable and stable steady states could be controlled with any one of the modes of control the proportional-integral-rate seeming equally adequate in both cases and better than proportional control.

If it is assumed that the proportional-integral-rate action would be used as the most suitable mode of control in all four systems, it is seen that the relative difficulty of control at the unstable steady state, as compared with the control at the stable steady state, is not as pronounced as might first be expected. The reason for this is that the autoregulatory nature of the stable steady state is not fast acting, and the system needs some help from a controller to reduce the settling time.

The control of the influent concentration was of the same order of difficulty as the control of influent temperature, while both were considerably more difficult to control than a disturbance in the inlet coolant temperature.

NOTATION

A = concentration of A in reactor
A_i = desired concentration of A
A_0 = inlet concentration of A
E = $\boldsymbol{\theta}_i - \boldsymbol{\theta}_0$, error signal
G = forward or open-loop transfer function
I_t = integral time
K = variable gain factor
k = reaction velocity coefficient
M = residue in pole at -0.00740
n = any positive or negative integer
Q = flow rate of reactant mixture through reactor, cu. ft./sec.
Q_c = coolant flow rate
R_t = rate (or derivative) time
S = Laplace transform variable
t = time, sec.
T = temperature of reactor
$\mathbf{T}$ = measured variable
T_c = average temperature of coolant
T_{ec} = inlet coolant temperature
T_i = desired temperature of reactor
T_0 = temperature of inlet reactant mixture
$\mathbf{V}$ = volume, cu. ft.

Greek Letters

α = $ha/2c_c\rho_c$
θ_i = input signal
θ_0 = output signal
ζ = damping coefficient
bold face or bar = transformed variable

Constants

a = area of cooling coil, 500 sq. ft.
A_s = concentration in reactor at steady state, 0.704 lb.-mole/cu. ft.
A_s' = concentration in reactor at steady state, 0.0589 lb.-mole/cu. ft.
A_{0s} = concentration in inlet mixture at steady state, 1.00 lb.-mole/cu. ft.
c = mean heat capacity of reactant mixture, 1.0 B.t.u./(lb.)(°R.)
c_c = mean heat capacity of coolant, 1.0 B.t.u./(lb.)(°R.)
ϵ = Activation energy, 44,700 B.t.u./lb.-mole
h = heat transfer coefficient, 100 B.t.u./(hr.)(sq. ft.)(°R.)
ΔH = heat of reaction, 20,000 B.t.u./lb.-mole
k_1 = reaction velocity constant, 2.7×10^{11} cu. ft./(lb.-mole)(sec.)
k_s = reaction velocity coefficient, 2.98×10^{-3} cu. ft./(lb.-mole)(sec.)
k_s' = reaction velocity coefficient, 1.354 cu. ft./(lb.-mole)(sec.)
Q_{cs} = steady state coolant flow rate, 0.289 cu. ft./sec.
Q_s = steady state reactant flow rate, 0.5 cu. ft./sec.
R = gas constant, 1.987 B.t.u./(lb.-mole)(°R.)
ρ = density of reactant mixture, 60.01 lb./cu. ft.
ρ_c = density of coolant, 62.4 lb./cu. ft.
T_{ecs} = steady state temperature of inlet coolant, 520°R.
T_{0s} = steady state temperature of inlet reactant mixture, 661.6°R.
T_s = steady state temperature of reactor, 700°R.
T_s' = steady state temperature of reactor, 859.5°R.
V = volume of reactor, 100 cu. ft.

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